$$\begin{cases} u_{tt} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) & 0 \le r < r_0,\ t>0 \\ u(r,0)=0 \\ u_t(r,0) = A \\ u(r_0,t)=0 \end{cases}$$

$$u = R(r)\cdot T(t)$$

$$\frac{T''}{a^2 T} = \frac{\frac{1}{r}(R' + rR'')}{R} = -\lambda$$

1) $\begin{cases} r^2R'' + rR' + \lambda r^2 R = 0 \\ R(r_0)=0 \end{cases}$ $n=0$

$$R = J_0\left(\frac{\mu_k^0 r}{r_0}\right)$$

$$\lambda = \left(\frac{\mu_k^0}{r_0}\right)^2, \quad \mu_k^0 : J_0(\mu_k^0)=0$$

2) $T'' = -\lambda a^2 T$

$$T_k = A_k\cos\left(\frac{a\mu_k^0}{r_0}t\right) + B_k\sin\left(\frac{a\mu_k^0}{r_0}t\right)$$

3) $u = \sum_{k=1}^{\infty} J_0\left(\frac{\mu_k^0 r}{r_0}\right)\left(A_k\cos\left(\frac{a\mu_k^0 t}{r_0}\right) + B_k\sin\left(\frac{a\mu_k^0 t}{r_0}\right)\right)$

$$u(r,0) = \sum_{k=1}^{\infty} J_0\left(\frac{\mu_k^0 r}{r_0}\right)\cdot A_k = 0 \Rightarrow A_k = 0$$

$$u_t\big|_{t=0} = \sum_{k=1}^{\infty} B_k \frac{a\mu_k^0}{r_0} J_0\left(\frac{\mu_k^0 r}{r_0}\right) = A$$

$$B_k\frac{a\mu_k^0}{r_0} = \frac{2}{r_0^2 (J_0'(\mu_k^0))^2}\cdot\int_0^{r_0} rAJ_0\left(\frac{\mu_k^0 r}{r_0}\right)dr = \left\{ z = \frac{\mu_k^0 r}{r_0}\right\} = \frac{2A}{r_0^2(J_0'(\mu_k^0))^2}\cdot\left(\frac{r_0}{\mu_k^0}\right)^2 z J_1($$

$$= \frac{2AJ_1(\mu_k^0)}{(J_0'(\mu_k^0))^2\mu_k^0} \Rightarrow B_k = \frac{2r_0 J_1(\mu_k^0)A}{a(\mu_k^0)^2(J_0'(\mu_k^0))^2}$$

$$\boxed{u(r,t) = \sum_{k=1}^{\infty} J_0\left(\frac{\mu_k^0 r}{r_0}\right)\cdot\sin\left(\frac{a\mu_k^0 t}{r_0}\right)\cdot\frac{2r_0J_1(\mu_k^0)A}{a(\mu_k^0)^2(J_0'(\mu_k^0))^2}}$$

$$\begin{cases} u_{tt} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + F & 0 \le r < r_0 \quad t > 0 \\ u(r,0) = 0 \\ u_t(r,0) = 0 \\ u_r(r_0,t) = A \end{cases}$$

$$u = v + w$$

w:
$$\begin{cases} w_{tt} = a^2 \frac{1}{r}(w_r + r w_{rr}) + F \\ u_r(r_0, t) = A \end{cases}$$

$$\exists\, w = Cr^2 + f(t)$$

$$f'' = a^2 \frac{1}{r}(2Cr + r\cdot 2C) + F$$

$$f'' = a^2 4C + F$$

$$2Cr_0 = A \Rightarrow C = \frac{A}{2r_0}$$

$$f'' = \frac{2a^2A}{r_0} + F$$

$$f = \left(\frac{2a^2A}{r_0} + F\right)\frac{t^2}{2}$$

$$w = \frac{A}{2r_0}r^2 + \left(\frac{2a^2A}{r_0} + F\right)\frac{t^2}{2}$$

v:
$$\begin{cases} v_{tt} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \\ v(r,0) = -\frac{A}{2r_0}r^2 \\ v_t(r,0) = 0 \\ v_r(r_0,t) = 0 \end{cases}$$

$$v = R(r)\cdot T(t)$$

$$\frac{T''}{a^2T} = \frac{\frac{1}{r}(R' + rR'')}{R} = -\lambda$$

1) $$\begin{cases} r^2R'' + rR' + \lambda r^2 R = 0 \\ R'(r_0) = 0 \end{cases}$$

$$R = J_0\left(\frac{\mu_k r}{r_0}\right)$$

$$\mu_k: \quad J_0'(\mu_k) = 0.$$

$$\lambda_k = \left(\frac{\mu_k}{r_0}\right)^2$$

$$\begin{cases} r^2R'' + rR' = 0 \\ R'(r_0) = 0 \end{cases}$$

$$\frac{1}{r}\left(\frac{d}{dr}(rR')\right) = 0$$

$$rR' = \text{const}$$

$$R = C_1 \ln r + C_2$$

$$R'(r_0) = C_1/r_0 = 0 \Rightarrow C_1 = 0$$

$$R = 1, \quad \lambda_0 = 0$$

2) $T_k'' + a^2\lambda_k T = 0$ $\qquad T_0'' = 0$

$T_k = A_k\cos(a\sqrt{\lambda_k}\, t) + B_k\sin(a\sqrt{\lambda_k}\, t)$ $\qquad T_0 = A_0 t + B_0$

3) $v(r,t) = \sum\limits_{k=1}^{\infty} J_0\left(\frac{\mu_k r}{r_0}\right)\left(A_k\cos\left(a\frac{\mu_k}{r_0}t\right) + B_k\sin\left(a\frac{\mu_k}{r_0}t\right)\right) + A_0 t + B_0$

$v_t(r,0) = \sum\limits_{k=1}^{\infty} J_0\left(\frac{\mu_k r}{r_0}\right) B_k\frac{a\mu_k}{r_0} + A_0 = 0 \Rightarrow B_k = 0 \quad A_0 = 0$

$v(r,0) = \sum\limits_{k=1}^{\infty} J_0\left(\frac{\mu_k r}{r_0}\right) A_k + B_0 = -\frac{Ar^2}{2r_0}$

$\|1\|^2 = \int_0^{r_0} r\,dr = \frac{r^2}{2}\Big|_0^{r_0} = \frac{r_0^2}{2}$

$\left\|J_0\left(\frac{\mu_k r}{r_0}\right)\right\|^2 = \frac{r_0^2(J_0(\mu_k))^2}{2}$

$B_0 = \frac{2}{r_0^2}\int_0^{r_0} r\cdot 1\cdot\left(-\frac{Ar^2}{2r_0}\right)dr = \frac{2}{r_0^2}\cdot\left(-\frac{A}{2r_0}\right)\cdot\frac{r_0^4}{4} = -\frac{Ar_0}{4}$

$A_k = \frac{2}{r_0^2(J_0(\mu_k))^2}\int_0^{r_0} r J_0\left(\frac{\mu_k r}{r_0}\right)\cdot\left(-\frac{Ar^2}{2r_0}\right)dr = -\frac{A}{r_0^3(J_0(\mu_k))^2}\int_0^{r_0} r^3 J_0\left(\frac{\mu_k r}{r_0}\right.$

$= -\frac{A}{r_0^3(J_0(\mu_k))^2}\cdot\int_0^{\mu_k}\left(\frac{r_0}{\mu_k}\right)^4 z^3 J_0(z)\,dz = \overbrace{-\frac{Ar_0}{(J_0(\mu_k))^2(\mu_k)^4}}^{C}\int_0^{\mu_k} z^3 J_0(z)\,dz =$

$= C\cdot\int_0^{\mu_k} z^2\, d(zJ_1(z)) = C\cdot z^2\cdot z\cdot J_1(z)\Big|_0^{\mu_k} - C\int_0^{\mu_k} 2z\cdot z J_1(z)\,dz =$

$= C\cdot\mu_k^3 J_1(\mu_k) - 2C\cdot z^2 J_2(z)\Big|_0^{\mu_k} = C\mu_k^3 J_1(\mu_k) - 2C\mu_k^2 J_2(\mu_k) =$

$= -\frac{Ar_0 J_1(\mu_k)}{(J_0(\mu_k))^2\mu_k} + \frac{2Ar_0 J_2(\mu_k)}{(J_0(\mu_k))^2\mu_k^2}$

Можно восп-ся ф-лами 3) $J_0'(x) = -J_1(x) \Rightarrow J_1(\mu_k) = -J_0'(\mu_k) = 0$

5) $J_2(x) = -J_0(x) + \frac{2\cdot 1\cdot J_1(x)}{x} \Rightarrow J_2(\mu_k) = -J_0(\mu_k) + 0$

Тогда $A_k = -\frac{2Ar_0}{J_0(\mu_k)\mu_k^2}$

$v(r,t) = -\frac{Ar_0}{4} - \sum\limits_{k=1}^{\infty} J_0\left(\frac{\mu_k r}{r_0}\right)\cdot\frac{2Ar_0\cos\left(a\frac{\mu_k}{r_0}t\right)}{\mu_k^2(J_0(\mu_k))}$

$u(r,t) = \frac{Ar^2}{2r_0} + \frac{t^2}{2}\left(\frac{2a^2A}{r_0} + F\right) +$
$-\frac{Ar_0}{4} - \sum\limits_{k=1}^{\infty} J_0\left(\frac{\mu_k r}{r_0}\right)\frac{2Ar_0\cos\left(a\frac{\mu_k}{r_0}t\right.}{\mu_k^2 J_0(\mu_k)}$